$$abc = 4Ar = 16p^3(p+1)(p-1)(2p^2-1)(4p^4+1)$$

and

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16A^2 = 1024p^6(p+1)^2(p-1)^2(2p^2-1)^2.$$

Inspired by the factorization $4p^4+1 = (2p^2+2p+1)(2p^2-2p+1)$, we let $a = 4p(2p^2-1)$, $b = 2p(p-1)(2p^2+2p+1)$, and $c = 2p(p+1)(2p^2-2p+1)$. Then

$$abc = 16p^3(p+1)(p-1)(2p^2-1)(4p^4+1)$$

as needed, so to complete the argument, it suffices to verify the second formula above. Letting $P = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$, we calculate

$$\begin{aligned} P &= (2p)^4(4p^3+4p^2-2p-2)(4p^3-4p^2-2p+2)(4p^2)(4p^2-4) \\ &= 1024p^6(p+1)(2p^2-1)(p-1)(2p^2-1)(p+1)(p-1) \\ &= 1024p^6(p+1)^2(p-1)^2(2p^2-1)^2. \end{aligned}$$

Hence the result holds for any integer $p > 1$. In particular, when $p = 2$, the triangle side lengths are 56, 52, and 60; when $p = 3$, the triangle side lengths are 204, 300, and 312.

Addendum: The sides of every Heronian triangle have the form $d(m+n)(mn-k-^2)$, $dm(n^2+k^2)$, and $dn(m^2+k^2)$, where m, n and k are positive integers with $\gcd(m,n,k) = 1$ and where d is a proportionality factor; see [1] for more details. Given any integer $p > 1$, we may take $m = p^2, n = p^2-1, k = p(p-1)$, and $d = \dfrac{2}{k}$ to produce the values of a, b, and c given above.

[1] https://en.wikipedia.org/wiki/Heronian-triangle

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, NY; Toshihiro Shimizu, Kawasaki Japan; David Stone and John Hawkins, Southern Georgia University; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Bazău, Romania, and the proposer.

- **5357:** *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

Determine all triangles whose side-lengths are positive integers (of which at least one is a prime number), whose semiperimeter is a positive integer, and whose area is equal to its perimeter.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let a, b, c be the positive integer side-lengths of a triangle, $p = \dfrac{a+b+c}{2}$ its semiperimeter and let us suppose that the area of that triangle, given by Heron's formula $\sqrt{p(p-a)(p-b)(p-c)}$ is equal to its perimeter $2p$.

Let $x = p-b,\ y = p-c,\ z = p-a$; then $xyz = (p-a)(p-b)(p-c) = 4p = 4(x+y+z)$ so $x = \dfrac{4(x+y)}{xy-4}$. By the triangle inequalities, x, y, z are positive integers so $xy = 4$ must be a positive integer as well. Without loss of generality, suppose that $a \le b \le c$; since $a = x+y, b = y+z,\ c = z+x$, and this is equivalent to $y \le x \le z$, so

$x+y \leq 2x \leq 2z = \dfrac{8(x+y)}{xy-4}$, from where $xy-4 \leq 8$; hence $y \leq \dfrac{12}{x} \leq \dfrac{12}{y}$ which implies $y \leq 3$, that is $y \in \{1,2,3\}$.

If $y=1$, then $x \leq \frac{1}{2}y = 12$ or equivalently $x \in \{1,2,3,4,5,6,7,8,9,10,11,12\}$. If $y=2$, then $2 = y \leq 12y = 6$, or what is the same $x \in \{2,3,4,5,6\}$ and if $y=3$, then $3 = y \leq x \leq 12y = 4$, or equivalently, $x \in \{3,4\}$.

From these possibilities the only ones that give positive integers for $z = \dfrac{4(x+y)}{xy-4}$ are $(x,y) = \{(5,1),\ (6,1),\ (8,1),\ (9,1),\ (3,2),\ (4,2),\ (6,2)\}$, which give

$$(a,b,c) = (x{+}y,\ y{+}z,\ z{+}x\} \in \{6,25,29),\ (7,15,20),\ (9,10,17),\ (10,9,17),\ (5,12,13),\ (6,8,10),\ (8,6,10)\}.$$

Thus, the triples of positive integer side-lengths of triangles whose area is equal to its perimeter are $(6,25,29), (7,15,20), (9,10,17), (5,12,13), (6,8,10)$ and since at least one of a,b,c is a prime number, we exclude the triple $(6,8,10)$ and since in all the other four cases the semiperimeter $p = \dfrac{a+b+c}{2}$ is a positive integer, the triangles we are looking for are those whose side lengths are $(6,25,29), (7,15,20), (9,10,17)$, or $(5,12,13)$. (Note also that only the last of them corresponds to a right triangle.)

Solution 2 and Comment by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The $A=P$ problem has a long history. In [2], Markov tells us that Dickson [1] attributes the solution to Whitworth and Biddle in 1904, then lists the only triangles with Area = Perimeter:

$$\begin{gathered}(6,8,10)\\(5,12,13)\\(6,25,29)\\(7,15,20)\\(9,10,17).\end{gathered}$$

Because our problem requires that one side be a prime, we see that the only solutions to the stated problem are the last four triangles above (note that each has an integral semiperimeter).

(The above result can probably now be considered as "common knowledge": it even appeared recently online on answers Yahoo.com [3]).

1. L. Dickson, History of the Theory of Numbers, Vol II, Dover Publications, Inc, New York, 2005 (reprint from the 1923 edition), p. 199.

2. L. P. Markov, Pythagorean Triples and the Problem A = mP for Triangles, Mathematics Magazine 79(2006) 114−121

3. From Dan, answers.Yahoo.com/question/index?qid=2081130185149AAua2RD, 7 years ago.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Jerry Chu (student, Saint George's School), Spokane, WA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton,

KS; David E. Manes SUNY College at Oneonta, Oneonta, NY; Ken Korbin, NewYork, NY; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

- **5358:** *Proposed by Arkady Alt, San Jose, CA*

Prove the identity $\displaystyle\sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1} = (r+1)^m(mr-1)+1.$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$$\begin{aligned}
(r+1)^m(mr-1)+1 &= \sum_{k=0}^{m} m\binom{m}{k}r^{k+1} - \sum_{k=1}^{m}\binom{m}{k}r^k \\
&= \sum_{k=1}^{m} m\binom{m}{k}r^{k+1} - \sum_{k=1}^{m-1}\binom{m}{k+1}r^{k+1} \\
&= mr^{m+1} + \sum_{k=1}^{m-1}\left(m\binom{m}{k} - \binom{m}{k+1}\right)r^{k+1} \\
&= mr^{m+1} + \sum_{k=1}^{m-1} k\binom{m+1}{k+1}r^{k+1} \\
&= \sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1}
\end{aligned}$$

where we have used that $m\binom{m}{k} - \binom{m}{k+1} = k\binom{m+1}{k+1}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

We have

$$(1+r)^m = \sum_{k=0}^{m}\binom{m}{k}r^k \tag{1}$$

and differentiating

$$mr(1+r)^{m-1} = \sum_{k=0}^{m} k\binom{m}{k}r^k. \tag{2}$$

Now

$$\begin{aligned}
\sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1} &= \sum_{k=2}^{m+1}(k-1)\binom{m+1}{k}r^k = \sum_{k=2}^{m+1} k\binom{m+1}{k}r^k - \sum_{k=2}^{m+1}\binom{m+1}{k}r^k \\
&\overset{(2),(1)}{=\!=} (m+1)r(1+r)^m - (m+1)r - (1+r)^{m+1} + 1 + (m+1)r \\
&= (r+1)^m(mr-1)+1.
\end{aligned}$$

Solution 3 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy

Proof Induction. Let $m = 1$. We have

$$\binom{2}{2} r^2 = (r+1)(r-1) + 1$$

which clearly holds.

Let's suppose it is true for $2 \le m \le n-1$. For $m = n$ we have

$$\sum_{k=1}^{m+1} k \binom{m+2}{k+1} r^{k+1} = (m+1)r^{m+2} + \sum_{k=1}^{m} k \left[\binom{m+1}{k+1} + \binom{m+1}{k} \right] r^{k+1} =$$

$$= (m+1)r^{m+2} + (r+1)^m(mr-1) + 1 \sum_{k=1}^{m} k \binom{m+1}{k} r^{k+1} \qquad (1)$$

$$\binom{m+2}{k+1} = \binom{m+1}{k+1} + \binom{m+1}{k}$$

and the induction hypothesis have been used. Moreover

$$\sum_{k=1}^{m} k \binom{m+1}{k} r^{k+1} \underbrace{=}_{k+1=p} r \sum_{p=0}^{m-1} (p+1) \binom{m+1}{p+1} r^{p+1} =$$

$$= r \sum_{p=1}^{m-1} p \binom{m+1}{p+1} r^{p+1} + r \sum_{p=0}^{m-1} \binom{m+1}{p+1} r^{p+1} =$$

$$= r \sum_{p=1}^{m} p \binom{m+1}{p+1} r^{p+1} - mr^{m+2} \underbrace{+}_{p+1=q} r \sum_{q=0}^{m+1} \binom{m+1}{q} r^q - r - r^{m+2}$$

The induction hypotheses and the Newton–binomial yield that it is equal to

$$r\left((r+1)^m(mr-1)+1\right) - mr^{m+2} + r(1+r)^{m+1} - r - r^{m+2}.$$

By inserting in (1) we get

$$\begin{aligned}
&(m+1)r^{m+2} + \left((r+1)^m(mr-1)+1\right)(r+1) - (m+1)r^{m+2} + r(1+r)^{m+1} - r = \\
&= (r+1)^{m+1}(mr-1) + (r+1) + r(1+r)^{m+1} - r = \\
&= (r+1)^{m+1}((m+1)r-1) - r(1+r)^{m+1} + (r+1) + r(1+r)^{m+1} - r = \\
&= (r+1)^{m+1}((m+1)r-1) + 1.
\end{aligned}$$

and the proof is complete.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Here we differentiate the given sum to get the Binomial Theorem, then integrate to get the desired sum.

Let $f(r) = \sum_{k=1}^{m} \binom{m+1}{k+1} r^{k+1} = \sum_{k=1}^{m} k \frac{(m+1)!}{(k+1)k(k-1)!(m+1-k-1)!} r^{k+1}$,

so,

$$\begin{aligned}
f'(r) &= \sum_{k=1}^{m} k \frac{(k+1)(m+1)!}{(k+1)k(k-1)!(m-k)!} r^k \\
&= \sum_{k=1}^{m} k \frac{(m+1)!}{(k-1)!(m-k)!} r^k \\
&= \sum_{k=0}^{m-1} mk \frac{(m+1)!}{(k-1)!(m-1-k)!} r^k, \text{ by reindexing} \\
&= m(m+1) \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-1-k)!} r^k, \\
&= m(m+1)r \sum_{k=0}^{m-1} \binom{m-1}{k} r^k \\
&= m(m+1)r(r+1)^{m-1} \text{ by the Binomial Theorem.}
\end{aligned}$$

Now we can integrate by parts to fine $f(r)$:

$$\begin{aligned}
f(r) &= \int m(m+1)(r(r+1)^{m-1} dr \\
&= m(m+1) \int r(r+1)^{m-1} dr \\
&= m(m+1) \left[\frac{1}{m} r(r+1)^m - \int \frac{1}{m}(r+1) dr \right] \\
&= m(m+1) \left[\frac{1}{m} r(r+1)^m - \frac{1}{m} \frac{(r+1)^{m+1}}{m+1} \right] + C \\
&= m(m+1) \left\{ \frac{(r+1)^m}{m} \frac{(mr-1)}{m+1} \right\} + C \\
&= (r+1)^m (mr-1) + C
\end{aligned}$$

Using the initial condition $f(0) = 0$ we find $C = 1$, so $f(r) = (r+1)^m(mr-1)+1$, as desired.

Editor's note: David and John also submitted a second solution to this problem that was similar to Solution 2 above.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Eastern Connecticut State University, Willimantic, CT David E. Manes SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

5359: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1} \le \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since $15a^3b+1$ can be represented as $(2a)^3\cdot\dfrac{15b+\dfrac{1}{a^3}}{8}$ then by AM-GM Inequality we obtain

$$\begin{aligned}\sum_{cyc}\sqrt[4]{15a^3b+1}=\sum_{cyc}\sqrt[4]{(2a)^3\cdot\frac{15b+\frac{1}{a^3}}{8}} &\le \sum_{cyc}\frac{3\cdot(2a)+\dfrac{15b+\frac{1}{a^3}}{8}}{4}\\ &= \sum_{cyc}\frac{48a+15b+\frac{1}{a^3}}{32}\\ &\le \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that

$$\sqrt[4]{11+15x^4} \le \frac{63}{32}x+\frac{1}{32x^3},\ x>0. \qquad (1)$$